The Relationship of Physical Applications of Fourier Transforms in Various Fields of Wave Theory and Circuitry*

E. FOLKE BOLINDER†

Summary—**A procedure is presented for connecting some known physical applications of Fourier transform pairs in different branches of the theory of waves and circuitry. After an investigation of the cases of diffraction, reflection, and coupling of waves, deflection of particles (which includes the cathode-ray-tube case and so-called gap effect) and the closely related scanning problem are examined. Finally, extension to random functions is discussed briefly.**

Introduction

DURING the last decades, more and more engineers and physicists have started to use the Laplace and Fourier transformations in solving their problems. The idea is that a problem that is difficult to solve in one domain might be easily solved after transformation to another domain, whereupon a transformation is made back into the original domain. Actually, every engineer is performing the same type of operations in multiplying two numbers by adding their logarithms on his sliderule.

In this paper, only the Fourier transformation arising from integration along the real axis in the two domains will be considered. The connection between this transformation, the Fourier transformation in the complex plane, the two-sided and the one-sided Laplace transformations has been lucidly described by van der Pol and Bremmer.[1] The method of extending real integrals into the complex plane is well known from the residue calculus. When even the complex integral is difficult to calculate, approximate integration has to be used. Until about eight years ago only two methods were known, both giving asymptotic series as results. The methods are called the saddlepoint method and the stationary phase method. Since that time, Cerrillo at M.I.T. has created a new theory of approximate integration founded on five new methods, all giving uniformly convergent series[2] as results.

* Manuscript received by the PGMTT, October 5, 1956. Reprinted from *Acta Polytechnica, Physics*, including *Nucleonics* Series, vol. 3, p. 189, 1956. The work was made possible by research grants from the Swedish Gov. Tech. Res. Council, Stockholm, Sweden, and Telefonaktiebolaget LM Ericsson, Stockholm, Sweden.

† Formerly with the Royal Inst. of Tech., Div. of Radio Eng., Stockholm, Sweden; now at the Res. Lab. of Electronics, Mass. Inst. of Tech., Cambridge, Mass.

[1] B. van der Pol and H. Bremmer, "Operational Calculus Based on the Two-Sided Laplace Integral," Cambridge University Press, London, Eng.; 1950.

[2] M. V. Cerrillo, "On the Evaluation of Integrals of the Type

$$f(\tau_1, \tau_2, \cdots, \tau_n) = \frac{1}{2\pi i}\int F(s)e^{W(s,\tau_1,\tau_2,\cdots,\tau_n)}ds$$

and the Mechanism of Formation of Transient Phenomena," Tech. Rep. No. 55, Res. Lab. of Electronics, Mass. Inst. Tech., Cambridge, Mass. Of six parts, no. 2a, "An Elementary Introduction to the Theory of the Saddlepoint Method of Integration," has been published; May 3, 1950.

In this paper, an elementary theory is formulated for connecting some known physical interpretations of Fourier transform pairs in different branches of the theory of waves and circuitry. This way of putting things may be considered a little unusual. Ordinarily, a specific problem is stated and the mathematical tools for solving it are looked for. Here, the tool, Fourier transformation, is given, and the connection between some known problems that can be solved by means of Fourier transforms is studied.

So many papers have been published on the use of Fourier transforms in engineering and physics that the author is compelled to refer only to certain specific papers in the different fields.

The Fourier Transformation

As is well known,[1] a Fourier series expansion may be written in the exponential form

$$\begin{cases} f(v) = \sum_{n=-\infty}^{\infty} a_n e^{j(2\pi n v/d)}, \text{ where} & (1) \\ a_n = \frac{1}{d}\int_{-d/2}^{d/2} f(v)e^{-j(2\pi n v/d)}dv. & (2) \end{cases}$$

By means of a limiting process, this series expansion can be transformed to an integral

$$f(v) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)e^{-j2\pi u y}dy\right]e^{j2\pi u v}du. \quad (3)$$

Separating:

$$\begin{cases} F(u) = \int_{-\infty}^{\infty} f(v)e^{-j2\pi u v}dv & (4) \\ f(v) = \int_{-\infty}^{\infty} F(u)e^{j2\pi u v}du. & (5) \end{cases}$$

The function $F(u)$ is called the Fourier transform of $f(v)$; $f(v)$ is called the inverse Fourier transform of $F(u)$. Together they form a Fourier pair. This Fourier pair, (4) and (5) has been tabulated; the most extensive table is the one by Campbell and Foster.[3]

[3] G. A. Campbell and R. M. Foster, "The practical application of the Fourier integral," *Bell Syst. Tech. J.*, vol. 7, pp. 639–707; October, 1928.

Campbell and Foster, "Fourier Integrals for Practical Applications," *Bell Syst. Tech. J.*, Monograph B584; September, 1931, and D. Van Nostrand Co., Inc., New York, N. Y., 1948.

When $f(v)$ and $F(u)$ are discontinuous, (4) and (5) have to be written as Fourier-Stieltjes integrals:

$$F(u) = \int_{-\infty}^{\infty} e^{-i2\pi uv} dg(v) \quad (6)$$

$$f(v) = \int_{-\infty}^{\infty} e^{i2\pi uv} dG(u) \quad (7)$$

where

$$g(v) = \int_{-\infty}^{v} f(\alpha) d\alpha \quad (8)$$

$$G(u) = \int_{-\infty}^{u} F(\beta) d\beta. \quad (9)$$

The functions $g(v)$ and $G(u)$ are called distribution functions; $f(\alpha)$ and $F(\beta)$ are called density distribution functions.

Very often the function $F(u)$ is "normalized":

$$F(u)_{\text{norm}} = \frac{F(u)}{F(u)_{u=0}} = \frac{\int_{-\infty}^{\infty} f(v) e^{-i2\pi uv} dv}{\int_{-\infty}^{\infty} f(v) dv}. \quad (10)$$

When the first derivatives of $f(v)$ or $F(u)$ are discontinuous, the so-called first corner theorem, stated by Cerrillo in a rigorous mathematical form,[4] has to be used. When, for simplicity, in the following we mainly treat continuous functions satisfying (4) and (5), it is understood that a small change of this Fourier pair makes it rigorously applicable as well to discontinuous functions and impulse functions.

Diffraction of Waves

Let us assume that we have a finite aperture in the interval $-(d/2) \leq x \leq (d/2)$, and that the aperture is excited by a wave, see Fig. 1, so that all elements of the aperture are in phase and have a density distribution function $f(x)$. We now select a point P in the so-called Fraunhofer region; *i.e.*, at such a distance from the origin that rays from the aperture towards P may be considered to be parallel. Adding the contributions of all element waves from the aperture at P we get

$$F_1(\theta, k) = \int_{-d/2}^{d/2} f(x) e^{i2\pi kx \sin \theta} dx \quad (11)$$

where $F_1(\theta, k)$ equals $F(\theta, k)$ multiplied by a phase factor determined by P's exact position, θ is the angle between the rays and a line perpendicular to the x axis, and the wave number $k = 1/\lambda$.

Putting

$$-k \sin \theta = u \quad (12)$$

[4] M. V. Cerrillo and E. F. Bolinder, "On Basic Existence Theorems in Network Synthesis, Part IV: Transmission of Pulses," Tech. Rep. No. 246, Res. Lab. of Electronics, M.I.T., Cambridge, Mass.; August, 1952.

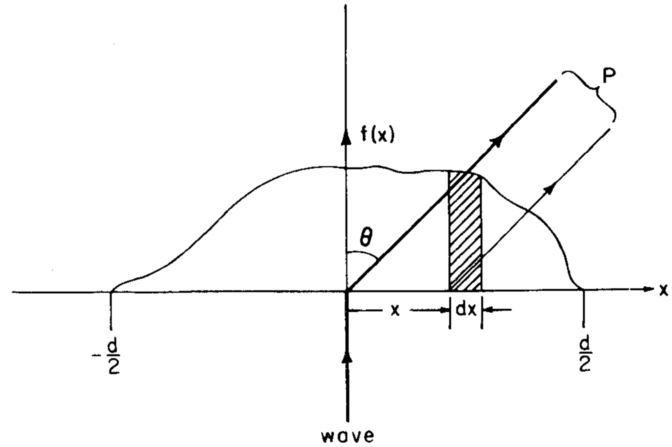

Fig. 1—Diffraction of a wave.

we get

$$F_1(u) = \int_{-d/2}^{d/2} f(x) e^{-i2\pi ux} dx. \quad (13)$$

This is a Fourier integral, because $f(x) = 0$, $|x| > (d/2)$. The inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} F_1(u) e^{i\pi zux} du. \quad (14)$$

We now have two cases to consider:

1) k is a constant, θ is a variable.

This case, known by Michelson in 1905, has been extensively treated in literature. A critical examination of the conditions under which it is right to consider $F_1(u)$ to be an antenna polar diagram was lately made by Booker and Clemmow.[5] They introduced a powerful concept called angular spectrum.

The integrals (13) and (14) can easily be extended to two or three dimensions, and have proved to be of great value in works dealing with diffraction of X rays and electrons by crystals. They have also been used, for example, in optics, in antenna theory, and in acoustics. The first to suggest the use of Fourier series in crystal analysis problems is thought to be W. H. Bragg in 1915. Since then, many papers have been published in that field; an article by Patterson[6] and a monograph by Wrinch[7] may be mentioned as examples.

During the second world war, Ramsay created an antenna theory[8] by interpreting the theorems in the Campbell-Foster table, mentioned above, in terms of antenna theory. At the same time Booker and his group

[5] H. G. Booker and P. C. Clemmow, "The concept of an angular spectrum of plane waves, and its relation to that of polar diagram and aperture distribution," *Proc. IEE*, part III, pp. 11–17; January, 1950.

[6] A. L. Patterson, "The diffraction of X-rays by small crystalline particles," *Phys. Rev.*, vol. 56, pp. 972–977; November, 1939.

[7] D. Wrinch, "Fourier transforms and structure factors," Axsred Monograph no. 2, The American Society of X-ray and Electron Diffraction, Murray Printing Co., Cambridge, Mass.; February, 1946.

[8] J. F. Ramsay, "Fourier transforms in the aerial theory," six parts in the *Marconi Rev.*, 1946–1948, based on two reports written in 1942–1943.

used Fourier transforms for determining cosec θ antenna patterns. On the basis of analogies in optics, Spencer[9] independently constructed an equivalent antenna theory. A book on the applications of Fourier integrals in optics has been published by Duffieux.[10]

In acoustics the Fourier pair, (13) and (14), has been used in designing microphones and loudspeakers.[11]

Both in electromagnetic theory and in acoustics, it is common to extend this Fourier pair to two dimensions and, after that, to make a transformation to polar coordinates. In that way, a Fourier-Bessel transform pair originates. For this pair the author has not been able to find any extensive table corresponding to the Campbell-Foster tables.

2) θ is a constant $= -90$ degrees, k is a variable.

In this case we get

$$\begin{cases} F(k) = \int_{-d/2}^{d/2} f(x) e^{-j2\pi kx} dx & (15) \\ f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx} dk & (16) \end{cases}$$

and the point P is located as shown in Fig. 2.

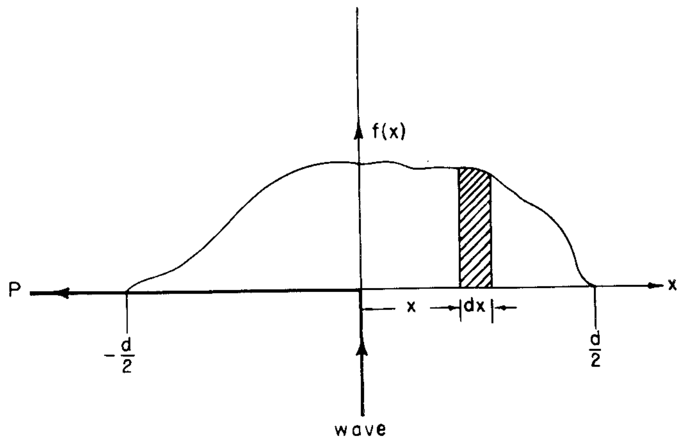

Fig. 2—Diffraction of a wave, $\theta = -90°$.

Reflection of Waves

In the diffraction case, Fig. 2, the wave exciting the aperture may be thought of as having its wavefront parallel to the x axis. If, instead, we assume that the wave is coming in toward the interval $-(d/2) \leq x \leq (d/2)$ in the positive x direction (Fig. 3) and is partly reflected in the interval, we get the following integral for the reflection coefficient at P:

$$\rho_1(k) = \int_{-d/2}^{d/2} P(x) e^{-j2\pi k 2x} dx \quad (17)$$

where $P(x)$ is the variable reflection coefficient in the interval. Since there are no reflections for $|x| > (d/2)$, the limits may be replaced by $-\infty$ and ∞ in the same way as in the diffraction case. The Fourier transform originating has the following mate:

$$P(x) = \int_{-\infty}^{\infty} \rho_1(k) e^{j2\pi k 2x} dk. \quad (18)$$

Since the wave travels back and forth, distance x in the diffraction case is now replaced by distance $2x$.

Practically, the simplest way of realizing this situation is by means of a dispersion-free coaxial tapered line. See Fig. 4. It can easily be shown that the connection

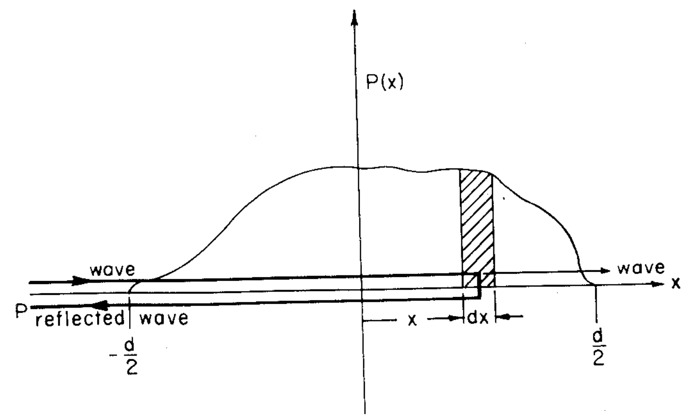

Fig. 3—Reflections in an inhomogeneous medium.

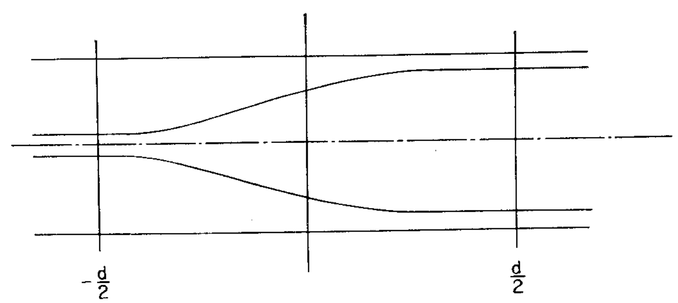

Fig. 4—Coaxial tapered line.

between the variable reflection coefficient $P(x)$, and the variable characteristic impedance, $Z_0(x)$, is the following:[12]

$$P(x) = \frac{1}{2} \frac{d \ln Z_0(x)}{dx} \cdot \quad (19)$$

Some restrictions have to be imposed on the reflection coefficients so that the Fourier integrals will be valid with great accuracy. These restrictions are that, in order to retain a plane field in the line, both $P(x)$ and $\rho_1(k)$ have to be small compared to unity.

[9] R. C. Spencer, "Fourier Integral Methods of Pattern Analysis," Rad. Lab. Rep. 762-1, M.I.T., Cambridge Mass.; PB 15305; January, 1946.

[10] P. M. Duffieux, "L'intégrale de Fourier et ses applications à l'optique," Société Anonyme des Imprimeries Oberthur, Rennes, France; 1946.

[11] H. F. Olsen, "Elements of Acoustical Engineering," D. Van Nostrand Co., Inc. New York. N.Y.; 1940.

[12] E. F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," *Trans. Royal Inst. Tech.* (Stockholm, Sweden), no. 48; 1951. Also see Proc. IRE, vol. 38, p. 1354; November, 1950.

Coupling of Waves

By exchanging the concept of reflection for that of coupling in the preceding section, we obtain an approximate theory of, for instance, the directional coupler.[13] The Fourier transform pair is directly

$$I_{\text{rev}}(k) = \int_{-d/2}^{d/2} I(x)e^{-j2\pi k2x}dx \quad (20)$$

$$I(x) = \int_{-\infty}^{\infty} I_{\text{rev}}(k)e^{j2\pi k2x}dk \quad (21)$$

where $I(x)$ is the coupled wave at a distance x, and I_{rev} is the total reversed wave in the coupled transmission line.

In the tapered line case the incident wave and the reflected element waves were enclosed in the same transmission line. In the coupling case the incident wave and the reflected waves travel in different transmission lines. See Fig. 5. Independent of the above theory, Miller, at the Bell Telephone Laboratories, recognized the Fourier transform formulation of the directivity of directional couplers. The work was presented in a joint paper with Mumford at the IRE Convention in 1951.[14]

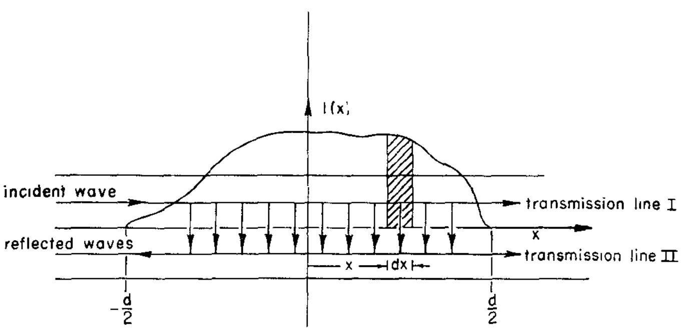

Fig. 5—Coupling of waves.

The directional couplers, both continuous and discrete, are well suited for frequencies in the microwave and vhf regions. For lower frequencies, it is possible to use a discrete folded directional coupler. See Fig. 6. Replacing the pieces of transmission lines between the coupling holes by delay networks and the couplings by amplifying devices, a distributed amplifier is obtained.

At low frequencies it is possible to use the upper half of the configuration in Fig. 6 and match it at the receiving end. See Fig. 7. The former coupling points are connected to phase-shifting attenuating or amplifying devices. Because of the low frequency, the tapping points T_1, T_2, $\cdots$, T_n may be connected together to a common output P. The type of filter originating was thoroughly investigated by Kallmann,[15] who called it a

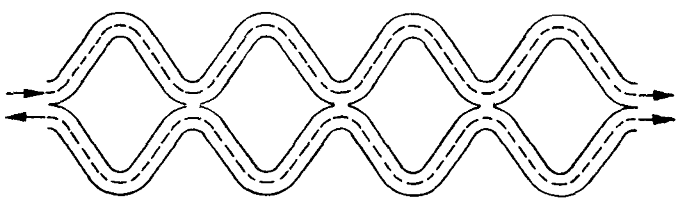

Fig. 6—A folded directional coupler.

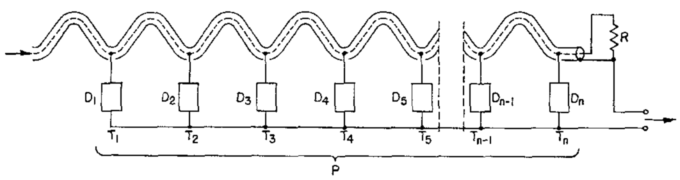

Fig. 7—Transversal filter.

"transversal" type of filter, to distinguish it from the ordinary "longitudinal" type of filter. The transversal filter has many features analogous to the grating spectroscope. Similar systems were described in patents filed in 1931 by Wiener and Lee.[16] By replacing the pieces of transmission lines in Fig. 7 by delay networks, Stutt[17] constructed a delay-line network with which transient phenomena, Fourier transforms, convolution integrals, and so on, can be studied on a cathode-ray tube.

At still lower frequencies the concept of distance completely loses its significance. By putting $x=ct$, $k=f/c$, where c is the velocity of light, we obtain the well-known Fourier pair in time-frequency domains:

$$F(f) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}dt \quad (22)$$

$$f(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft}df. \quad (23)$$

Refraction and Polarization of Waves

For the sake of completeness, it would have been nice if the concept of reflection given above could also be exchanged for the concepts of refraction or polarization. However, as shown by Fig. 8, in these cases no interference is obtained between element waves in the same sense as in the other cases, so that no Fourier transform pairs similar to the ones above are obtained.

Deflection of Particles: The Scanning Problem

In the different cases above, we have assumed that the point P is fixed, and that the waves move and are added at P. We may, however, just as well assume that we have a fixed field varying sinusoidally with time in the interval $-(d/2)\leq x\leq(d/2)$, and that P con-

[13] E. F. Bolinder, "Approximate theory of the directional coupler," Proc. IRE, vol. 39, p. 291; March, 1951.

[14] S. E. Miller and W. W. Mumford, "Multi-element directional couplers," presented before the IRE National Convention, New York, N. Y.; March, 1951.

[15] H. E. Kallmann, "Transversal filters," Proc. IRE, vol. 28, pp. 302–310; July, 1940.

[16] N. Wiener and Y. W. Lee, "Electrical network system," U.S. Patent No. 2024900; December 17, 1935 (filed September 2, 1931).

[17] C. A. Stutt, "Experimental Study of Optimum Filters," Tech. Rep. No. 182, Res. Lab. of Electronics, M.I.T., Cambridge, Mass.; May, 1951.

stitutes a particle moving through the interval. The field may be transversal or longitudinal and the particle may be of different kinds. During its travel through the field, the particle is exposed to influences from the field, and the influences are summed up after the particle has passed through the interval.

Because the particle P always travels with a velocity less than that of light, we have to introduce a fictitious wavelength $\lambda_e = 1/k_e$, if we want to use the same formulas as those given above. If v is the velocity of the particle P, and c is the velocity of light,

$$k_e = \frac{c}{v} k. \tag{24}$$

The Fourier transform pair is

$$F_1(k_e) = \int_{-d/2}^{d/2} f(x) e^{-j2\pi k_e x} dx \tag{25}$$

$$f(x) = \int_{-\infty}^{\infty} F_1(k_e) e^{j2\pi k_e x} dk_e. \tag{26}$$

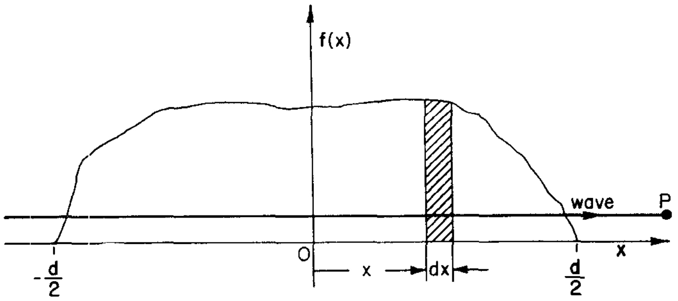

Fig. 8—A wave through a refracting or polarizing medium.

The Cathode-Ray-Tube Case

If the particle P constitutes an electron, and the density distribution function $f(x)$ is an electric field transverse to the x axis (formed, for instance, by metallic plates according to Fig. 9), then the cathode-ray-tube case is obtained. It has been shown[18] that in this case $F_1(k_e)$ is the dynamic sensitivity factor of the cathode-ray tube. Theoretically, the case may be thought of as originating from the reflection case by exchanging the concept of reflection for that of angular deflection. The corresponding approximations are that, both the variable angular deflection of the electron beam, $f(x) = \phi(x)$, and the total deflection angle at the end of the deflection, $F_1(k_e) = \phi_d(k_e)$, must be small.

The Gap (Slit) Effect

If an electron P, instead of passing through a transverse electric field, passes through a parallel field (see, for example, Fig. 10), the instantaneous velocity of P will be changed. If we assume that the velocity change

[18] E. F. Bolinder, "A theory of determining the dynamic sensitivity of cathode-ray-tubes at very high frequencies by means of Fourier transforms," IRE Trans., vol. ED-2, pp. 44–50; January, 1955.

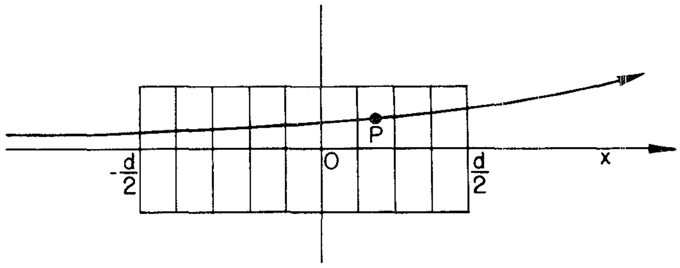

Fig. 9—An electron in a transversal field.

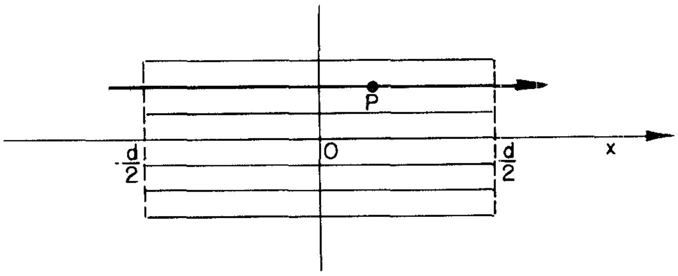

Fig. 10—An electron in a longitudinal field.

is small, so that the total transit time through the field can be assumed to be constant, then $F_1(k_e)$ constitutes the so-called beam coupling factor obtained, for instance, in klystron theory. An analogous transit time factor is obtained in transistor theory. The coining of the expression "gap effect" for these factors perhaps is still better understood in cases in which P constitutes a particle of a band running in front of a gap having a fixed field distribution. Examples are magnetic tape recording[19] and reproduction of movie film.[20,21] In all of the examples given above, the gap factor, which for a parallel slit of width d follows the well-known expression

$$F_1(k_e) = \frac{\sin \pi k_e d}{\pi k_e d} \tag{27}$$

may be changed, inside certain limits, by changing the field distribution in the slit.

The Scanning Problem

If, instead of a moving point P and a fixed aperture, we have a fixed point P and a moving aperture, the conditions of the scanning problem are fulfilled. Two fundamental papers dealing with the scanning problem in two dimensions have been written by Mertz and Gray[22] and Wheeler and Loughren.[23] Both papers are

[19] F. Krones, "Die Magnetische Schallaufzeichnung in Theorie und Praxis," special edition of Radiotechnik, *Z. für Hochfrequenz.* Verlag B Erb, Wien, 1952.

[20] W. Meyer-Eppler, "Tonfilmspalt und filmfrequenzgang," *Kino-Technik*, part I, pp. 1–7; January, 1943; part II, pp. 16–18; February, 1943.

[21] W. Meyer-Eppler, "Verzerrungen, die durch die endliche durchlassbreite physikalischer apparate hervorgerufen werden, nebst Anwendung auf die Periodenforschung," *Ann. Phys.*, vol. 41, pp. 261–300; April, 1942.

[22] P. Mertz and F. Gray, "A theory of scanning and its relation to the characteristics of the transmitted signal in telephotography and television," *Bell Syst. Tech. J.*, vol. 13, pp. 464–515; July, 1934.

[23] H. A. Wheeler and A. V. Loughren, "The fine structure of television images," Proc. IRE, vol. 26, pp. 540–575; May, 1938.

concentrated on the scanning problem in television, but the latter paper stresses the analogous conditions in optics.

The step from the slit problem in optics to optical diffraction of an aperture is very small. Thus we may say that we have returned to our starting point.

Fourier Pairs in Time and Frequency Domains

In our modern world of pulse-modulated links, television sets, and radars, the Fourier pair in time and frequency domains, (22) and (23), known since the days of Fourier and Lord Rayleigh, has had extensive use. The author confines himself to pointing out two papers by Cherry[24] and Levy.[25]

This Fourier pair lately obtained special application in the theory of superregeneration.[26–29]. In this theory

$$f(t) = s(t) = \exp \int_0^t \alpha d\tau \qquad (28)$$

where $s(t)$ represents the variation in impulse sensitivity with time and is, therefore, called "sensitivity pulse," $\alpha = G/2C$, G is the conductance, and C is the capacitance of a single tuned circuit.

Wheeler,[30] in 1942, recognized that the frequency function representing the selectivity of a simple superregenerative circuit is the Fourier transform of the sensitivity pulse.

Random Functions

The different Fourier transform pairs discussed above can be extended to be valid for random functions. The well-known Wiener-Khintchine theorem states that the power density spectrum and the correlation function are Fourier transforms of each other. Correlation functions have found applications, for example, in X-ray crystallography (Patterson diagrams), optics, antenna theory, acoustics, communication theory, and statistics. Much work has been done and is still going on, to figure out the exact interrelations among these different fields. A complete treatment of the random functions cases with references is, however, outside the scope of this paper.

Conclusion

The different physical applications of Fourier transform pairs discussed above show a selection of the great variety of problems which can be treated by one and the same mathematical tool, Fourier transformation theory. Thus, results in one field are immediately applicable in another field. Computations are simplified by existing tables and calculating machines. In the cases in which severe assumptions limit the practical use of the theory, the Fourier transform pairs have to be modified. Even in these idealized cases, however, the Fourier theory is of great value because it is simple and comprehensible. The presentation above constitutes an attempt to show, in a graphic way, how some known Fourier transform pair applications may be thought of as fitting together.

Acknowledgment

The work presented above was started in the summer of 1954, when the author was at the Royal Institute of Technology, Stockholm, Sweden; it was continued while he was a guest at Instituto Nacional de la Investigacion Cientifica, Mexico D. F., Mexico, during the winter of 1954–55; and it was completed at the Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., in the summer of 1955. The author expresses his gratitude to the Swedish Government Technical Research Council, Stockholm, Sweden, and Telefonaktiebolaget LM Ericsson, Stockholm, Sweden, for grants which made the Mexico-United States trip possible. He also takes great pleasure in thanking Dr. Manuel V. Cerrillo for his constructive criticism in reading the first part of the paper.

[24] E. C. Cherry, "Pulse response: a new approach to ac electric network theory and measurement," *J. IEE*, vol. 92, part III, pp. 183–196; September, 1945.
[25] M. M. Levy, "Fourier series and Fourier transform analysis," *J. Brit. IRE*, part I, vol. 6, pp. 64–73; March-May, 1946; part II, vol. 6, pp. 228–246; December, 1946.
[26] H. A. Wheeler, "A simple theory and design formulas for superregenerative receivers," Wheeler Monographs No. 3; June, 1948.
[27] H. A. Wheeler, "Superselectivity in a superregenerative receiver," Wheeler Monograph No. 7; November, 1948.
[28] W. E. Bradley, "Superregenerative detection theory," *Electronics*, vol. 21, pp. 96–98; September, 1948.
[29] A. Hazeltine, D. Richman, and B. D. Loughlin, "Superregenerative design," *Electronics*, vol. 21, pp. 99–102; September, 1948.
[30] H. A. Wheeler, private correspondence.